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# Principal Loop Bundles: Toward Nonassociative Gauge Theories

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We introduce a nonassociative gauge field theory with nonassociative symmetries. The approach is based on the nonassociative generalization of principal bundles theory.

## 1. PRELIMINARIES

Difficulties in unifying all interactions prompt us to look for a mathematical structure beyond groups. The quantum group approach is a construction of this type where the Lie group symmetry is replaced by a quantum group symmetry and the latter reduces to the standard one in some limit (Đurđevich, 1996, 1997, 2000). Another possibility is a nonassociative generalization of a Lie group, such as quasigroups and smooth loops.

We developed gauge theories based on a nonassociative generalization of principal bundles theory (Nesterov, 1989, 1999, 2000; Nesterov and Stepanenko, 1986). The algebraic theory of quasigroups and loops may be found in Belousov (1967), Bruck (1971), Pflugfelder (1990), Chein *et al.* (1990), and Sabinin (1999).

Let $\langle Q, \cdot\rangle$ be a groupoid with a binary operation $(a, b) \mapsto a \cdot b$. A groupoid $\langle Q, \cdot\rangle$ is called a *quasigroup* if equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions: $x = a\backslash b$, $y = b/a$. A *loop* is a quasigroup with a two-sided identity $a \cdot e = e \cdot a = a$, $\forall a \in Q$. A loop $\langle Q, \cdot, e\rangle$ that is also a differential manifold and an operation $\phi(a, b) := a \cdot b$ is a smooth map is called a *smooth loop*. We define

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$$L_a b = R_b a = a \cdot b, \qquad l_{(a,b)} = L_{a\cdot b}^{-1} \circ L_a \circ L_b, \qquad \hat{l}_{(a,b)} = L_{a\cdot b} \circ l_{(a,b)} \circ L_{a\cdot b}^{-1} \tag{1}$$

where $L_a$ is a *left translation*, $R_b$ is a *right translation*, $l_{(a,b)}$ is a *left associator*, and $\hat{l}_{(a,b)}$ is an adjoint associator,

$$\hat{l}_{(a,b)} = L_a \circ L_b \circ L_{a\cdot b}^{-1}$$

The operation $a \cdot b$ need not be associative, $a \cdot (b \cdot c) \neq (a \cdot b) \cdot c$, but there exists quasiassociativity:

(a) The left identity of quasiassociativity

$$a \cdot (b \cdot c) = (a \cdot b) \cdot l_{(a,b)} c$$

where $l_{(a,b)}$ is the (left) associator defined above.

(b) The right identity of quasiassociativity

$$r_{(b,c)}\, a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

where $r_{(b,c)} = R_{(b,c)}^{-1} \circ R_c \circ R_b$ is the right associator.

Let $T_e(Q)$ be the tangent space of $Q$ at the neutral element $e$. Then for each $X_e \in T_e(Q)$, we construct a smooth vector field on $Q$

$$X_b = L_{b*} X_e, \qquad b \in Q, \quad X_e \in T_e(Q), \quad X_b \in T_b(Q)$$

where $L_{b*}$: $T_e(Q) \mapsto T_b(Q)$ denotes the differential of the left translation. Note that $X_b$ satisfies

$$L_{a*} X_b = \hat{l}_{(a,b)*} X_{a\cdot b}, \qquad \forall a, b \in Q$$

*Definition 1.1.* A vector field $X$ on $Q$ which satisfies the relation $L_{a*} X_b = \hat{l}_{(a,b)*} X_{a\cdot b}$ for any $a, b \in Q$ is called a *left fundamental* or *left quasiinvariant* vector field.

In the view of the noncommutativity of the right and left translations, $L_a \circ R_b \neq R_b \circ L_a$, the definitions $Ad_g g' = L_g \circ R_g^{-1} g'$ and $\tilde{A}d_g g' = R_g^{-1} \circ L_g g'$ are not equivalent. We define a generalized adjoint map of $Q$ on itself in the following way.

*Definition 1.2.* A map

$$Ad_b(a) = L_a^{-1} \circ R_b^{-1} \circ L_{a\cdot b}: \quad Q \mapsto Q \tag{2}$$

is called an *Ad-map*.

*Remark 1.1.* $Ad_b(e) = R_b^{-1} \circ L_b$.

The Ad-map (2) generates the map

$$\mathsf{Ad}_b(a) := (Ad_b(a))_* = L^{-1}_{a*}R^{-1}_{b*}L_{a\cdot b*}: \quad T_e(Q) \mapsto T_e(Q)$$

*Definition 1.3.* A vector-valued 1-form $\boldsymbol{\omega}$ is said to be *canonical* Ad-form if is defined $\forall a \in Q$ through the relation

$$\omega(V_a) = V_e, \quad V_e = L^{-1}_{a*}V_a, \quad V_e \in T_e(Q), \quad V_a \in T_a(Q)$$

and $V \in T(Q)$ is a left fundamental vector field.

*Theorem 1.1* (Nesterov, 1989, 1999). The canonical form $\boldsymbol{\omega}$ is a left fundamental form and it is transformed under left (right) translations as

$$(L^*_b\omega)V_a = l_{(b,a)*}\omega(V_a), \qquad (R^*_b\omega)V_a = \mathsf{Ad}^{-1}_b(a)\omega(V_a)$$

## 2. PRINCIPAL LOOP BUNDLES

Let $M$ be a manifold and $\langle Q, \cdot, e\rangle$ a smooth two-sided loop. A *principal loop Q-bundle* (a principal bundle with the structure loop $Q$) is a triple $(P, \pi, M)$, where $P$ is a manifold and the following conditions hold:

1. $Q$ acts freely on $P$ by the right map: $(p, a) \in P \times Q \mapsto \tilde{R}_a p \equiv pa \in P$.
2. $M$ is the quotient space of $P$ by the equivalence relation induced by $Q$, $M = P/Q$, and the canonical projection $\pi: P \to M$ is a smooth map onto.
3. $P$ is *locally trivial*, that is, for each $x \in M$, there exists an open neighborhood $U$ and a diffeomorphism $\Phi: \pi^{-1}(U) \mapsto U \times Q$ such that for any point $u \in \pi^{-1}(U)$, it has the form $\Phi(u) = (\pi(u), \varphi(u))$, where $\varphi$ is the map from $\pi^{-1}(U)$ to $Q$ satisfying $\varphi(\tilde{R}_a p) = R_a\varphi(p)$.

We say $P$ is a *total space* or *bundle space*, $Q$ is a *structure loop*, $M$ is a *base* or *base space*, and $\pi$ is a *projection*. For any $x \in M$, the inverse image $\pi^{-1}(x)$ is the $F_x$ *fiber* over $x$. Any fiber is diffeomorphic to $Q$ and the loop $Q$ acts transitively on each fiber, satisfying the right identity quasiassociativity

$$\tilde{R}_{a\cdot b}p = \tilde{R}_a \circ \tilde{R}_b\tilde{r}(a, b)p, \qquad \varphi(\tilde{r}_{(a,b)}p) = r_{(a,b)}\varphi(p)$$

Let us consider a covering $\{U_\alpha\}$ of $M$, which can be chosen in such a way that the restriction of the fibration to each open set $U_\alpha$ is trivializable. This implies that there exists a diffeomorphism $\Phi_a: \pi^{-1}(U_\alpha) \mapsto U_\alpha \times Q$. The set $\{U_\alpha, \Phi_\alpha\}$ is called a *local trivialization*.

*Proposition 2.1* (Nesterov, 1999). The right map on the fiber does not depend on the trivialization: $\Phi^{-1}_\alpha(\pi(p), \varphi_\alpha(\tilde{R}_q p)) = \Phi^{-1}_\beta(\pi(p), \varphi_\beta(\tilde{R}_q p))$.

*Definition 2.1.* The family of maps $\{q_{\beta\alpha}(p) = R^{-1}_{q_\alpha} q_\beta \colon \pi^{-1}(U_\alpha \cap U_\beta) \mapsto Q\}$, where $q_\alpha := \varphi_\alpha(p)$, $q_\beta := \varphi_\beta(p)$, $p \in \pi^{-1}(U_\alpha \cap U_\beta)$, is called the family of *transition functions* of the bundle $P(M, Q)$ corresponding to the covering $\{U_\alpha\}$ on $M$.

*Proposition 2.2.* Transition functions $q_{\beta\alpha}(p)$ change under right translations as

$$R_a q_{\beta\alpha}(p) = r(a, q_\alpha) q_{\beta\alpha}(p)$$

where $r(a, q_\alpha)$ is the right associator.

*Definition 2.2.* Let $\{U_\alpha, \Phi_\alpha\}$ be the local trivialization and $e \in Q$ a neutral element; then $\sigma_\alpha = \Phi_\alpha^{-1}(x, e)$, $x \in U_\alpha$, is called the *local section* associated with this trivialization.

Let $u \in P$; then one can describe the local section as follows: $\sigma_\alpha = \tilde{R}^{-1}_{q_\alpha(u)} u$, where by definition, $q_\alpha(u) = \varphi_\alpha(u)$.

*Proposition 2.3.* The section $\sigma_\alpha$ does not depend on the choice of the point in the fiber.

*Proof.* Let $u, p \in P$ and $u = \tilde{R}_{q_\alpha(u)} \sigma_\alpha$, $p = \tilde{R}_{q_\alpha(p)} \sigma'_\alpha$. There exists an element $a \in Q$ such that $p = \tilde{R}_a u$. Then we have

$$\begin{aligned}\Phi_\alpha(\tilde{R}_{q_\alpha(p)} \sigma'_\alpha) &= \Phi_\alpha(p) = \Phi_\alpha(\tilde{R}_a u) = \Phi_\alpha(\tilde{R}_a \circ \tilde{R}_{q_\alpha(u)} \sigma_\alpha) \\ &= (\pi(p), R_a \circ R_{q_\alpha(u)} \varphi_\alpha(\sigma_\alpha)) = (\pi(p), R_a \circ R_{q_\alpha(u)} e) \\ &= \Phi_\alpha(\tilde{R}_{q_\alpha(u)\cdot a} \sigma_\alpha)\end{aligned}$$

Taking into account $q_{\alpha(p)} = q_{\alpha(u)} \cdot a$, we obtain $R_{q_\alpha(p)} \sigma'_\alpha = R_{q_\alpha(p)} \sigma_\alpha$. This gives $\sigma_\alpha = \sigma'_\alpha$ and, hence, the section $\sigma_\alpha$ does not depend on the point of the fiber, while the dependence on the base, $\sigma_\alpha = \sigma_\alpha(x)$, holds. ■

*Proposition 2.4.* Let the point $x$ lie in the intersection of the neighborhoods $U_\alpha$ and $U_\beta$, $x \in U_\alpha \cap U_\beta$; then the following formula holds:

$$\sigma_\alpha(x) = \tilde{R}_{q_{\beta\alpha}} \sigma_\beta(x) \tag{3}$$

*Proof.* Let $u = \tilde{R}_{q_\alpha} \sigma_\alpha$; rewriting this formula in the chart $U_\beta$, we obtain

$$\begin{aligned}\Phi_\beta(\tilde{R}_{q_\alpha} \sigma_\alpha) &= \Phi_\beta(\tilde{R}_{q_\beta} \sigma_\beta) = (\pi(u), R_{q_\beta} \varphi_\beta(\sigma_\beta)) = (\pi(u), R_{q_{\beta\alpha} \cdot q_\alpha} \varphi_\beta(\sigma_\beta)) \\ &= (\pi(u), R_{q_\alpha} \circ R_{q_{\beta\alpha}} e) \\ &= (\pi(u), R_{q_\alpha} \circ R_{q_{\beta\alpha}} \varphi_\beta(\sigma_\beta)) = \Phi_\beta(\tilde{R}_{q_\alpha} \circ \tilde{R}_{q_{\beta\alpha}} \sigma_\beta)\end{aligned}$$

where the relation $\varphi_\beta(\sigma_\beta) = e$ has been used. Finally, we obtain $\sigma_\alpha(x) = \tilde{R}_{q_{\beta\alpha}} \sigma_\beta(x)$ in the intersection $U_\alpha \cap U_\beta$. ■

*Example 2.1.* Principal $QU(1)$ bundle over $S^2$. We define a smooth loop $QU(1)$ as a loop of multiplication by unimodular complex numbers, with elements $e^{i\alpha}$, $0 \leq \alpha < 2\pi$, and operation

$$e^{i\alpha} * e^{i\beta} = e^{i(\alpha \dotplus \beta)} \tag{4}$$

where $\alpha \dotplus \beta = \alpha + \beta + F(\alpha, \beta)$, and $F(\alpha, \beta)$ is a smooth function such that $F(\alpha, \beta) = F(\beta, \alpha)$, $F(\alpha, 0) = F(0, \beta) = 0$. Keeping in mind only the nonassociative case of the operation (4), we assume further that $F(\alpha, \beta) + F(\alpha \dotplus \beta, \gamma) \neq F(\beta, \gamma) + F(\alpha, \beta \dotplus \gamma)$.

We construct the bundle by taking

$$\text{Base } M = S^2 \qquad \text{with coordinates} \quad 0 \leq \theta < \pi, \quad 0 \leq \varphi < 2\pi$$

$$\text{Fiber } QU(1) = S^1 \qquad \text{with coordinates } e^{i\alpha}$$

We break $S^2$ into two hemispheres $H_\pm$ with $H_+ \cap H_-$ being a thin strip parametrized by the equatorial angle $\varphi$. Locally the bundle looks like

$$H_- \times QU(1) \qquad \text{with coordinates} \quad (\theta, \varphi, e^{i\alpha_-})$$

$$H_+ \times QU(1) \qquad \text{with coordinates} \quad (\theta, \varphi, e^{i\alpha_+})$$

In $\pi^{-1}(H_- \cap H_+)$ the elements $e^{i\alpha_+}$ and $e^{i\alpha_-}$ must be related by the transition function $e^{i\gamma}$:

$$e^{i\alpha_+} = e^{i\gamma} * e^{i\alpha_-} = e^{i(\gamma \dotplus \alpha_-)}$$

This implies

$$\alpha_+ = \alpha_- + \gamma + F(\gamma, \alpha_-)$$

Taking into account that the resulting structure must be a manifold, we find that

$$\gamma + F(\gamma, \alpha_-) = n\varphi \tag{5}$$

where $n$ *must be integer.*

*Comment 2.1.* The structure of the obtained manifold $P$ is unknown.

## 3. CONNECTION, CURVATURE, AND BIANCHI IDENTITIES

Let $P(M, Q)$ be the principal loop $Q$-bundle over the manifold $M$. For any $u \in P$ a tangent space at $u$, we denote as $T_u(P)$ (or simply $T_u$) and the tangent to the fiber passing through $u$ as $\mathcal{V}_u$. We call $\mathcal{V}_u$ a *vertical subspace*. It is generated by the right translations on the fiber: $u \mapsto \tilde{R}_a u$, $a \in Q$, $u \in P$:

$$X_u = \frac{d}{dt} \tilde{R}_{a(t)} u\big|_{t=0}, \qquad X_u \in \mathcal{V}_u, \quad a \in Q, \quad u \in P \tag{6}$$

The basic idea of connection is to compare the points in the “neighboring” fibers in a way that is not dependent on a local trivialization. Let $\gamma(t) \in M$ be a smooth curve. A *horizontal lift* of $\gamma$ is a curve $\tilde{\gamma}(t) \in P$ such that $\pi(\tilde{\gamma}(t)) = \gamma(t)$. Evidently, for determining $\tilde{\gamma}(t)$, it is suffucient to define at any point of it a tangent vector $\tilde{X}$: $\pi_* \tilde{X} = X$, where $X$ is the tangent vector to $\gamma(t)$. A set $\{\tilde{X}\}$ is called a *horizontal subspace* $\mathcal{H}_u$.

*Definition 3.1.* A *connection form* on a principal $Q$-bundle is a vector-valued 1-form taking values at $T_e(Q)$, which satisfies:

(i) $\omega(X_p) = X_e$, where $X_p \in \mathcal{V}_p$, $X_e \in T_e(Q)$ are determined according to (7), (8).

(ii) $(\tilde{R}_a^* \omega)\, X_p = \mathrm{Ad}_a^{-1}(q)\omega(X_p)$, where $q = \varphi(p)$.

(iii) The horizontal subspace $\mathcal{H}_p$ is defined as a kernel of $\omega$:

$$\mathcal{H}_p = \{X_p \in T_p(P) \colon \omega(X_p) = 0\}$$

This definition implies that a connection in $Q$-bundle is determined by the left canonical Ad-form. The connection allows us to decompose any vector $Z \in T_p(P)$ in the form $Z = X + Y$, where $X = horZ \in \mathcal{H}_p$ is the horizontal component of the vector $Z$ and $Y = verZ \in \mathcal{V}_p$ is the vertical one. The map $\varphi$ induces the map of the vertical subspace $\mathcal{V}_p$ onto the tangent space to $Q$, $\mathcal{V}_p \overset{\varphi_*}{\to} T_q(Q)$, $q = \varphi(p)$:

$$V_p = \frac{d}{dt} \tilde{R}_{a(t)} p \Big|_{t=0} \overset{\varphi_*}{\to} \hat{V}_q = \frac{d}{dt} R_{a(t)} q \Big|_{t=0} \tag{7}$$

Note that the vector field $\hat{V}_q$ is left quasiinvariant. Indeed, it can be written as $\hat{V}_q = (L_q)_* V_e$, where

$$V_e = \frac{da(t)}{dt}\Big|_{t=0} \in T_e(Q) \tag{8}$$

*Definition 3.2.* Let $V_e \in T_e(Q)$. The vector field $V_p \in \mathcal{V}_p$ connecting with $V_e$ by means of (7), (8) is called a *fundamental* vector field.

With the given connection form, a local 1-form taking values in $T_e(Q)$ can be associated as follows. Let $\sigma\colon U \subset M \mapsto \sigma(U) \subset P$, $\pi \circ \sigma = \mathrm{id}$, be a local section of a $Q$-bundle $Q \mapsto P \mapsto M$ which is equipped with a connection 1-form $\omega$. Define the *local $\sigma$-representative* of $\omega$ to be the vector-valued 1-form [taking values a $T_e(Q)$] $\omega^U$ on the open set $U \subset M$ given by $\omega^U := \sigma^* \omega$.

*Theorem 3.1.* (On reconstruction of the connection form). For a given canonical 1-form $\tilde{\omega}$ defined on $U \subset M$ with values in $T_e(Q)$ and the section $\sigma\colon U \mapsto \pi^{-1}(U)$, there exists one and only one connection 1-form $\omega$ on $\pi^{-1}(U)$ such that $\sigma^* \omega = \tilde{\omega}$.

*Proof.* Let $p_0 = \sigma(x)$ and $Z \in T_{p_0}(P)$. We have $Z = X_1 + X_2$, where $X_1 := (\sigma_* \circ \pi_*)Z$ and $X_2 \in \mathcal{V}_{p_0}$, $\pi_* X_2 = 0$. Define at $p_0$ the 1-form $\omega$ to be the vector-valued 1-form given by $\omega_{p_0} = \tilde{\omega}_x(\pi_* X) + \hat{X}_2$. Continuation of the 1-form $\omega$ onto all points of the fiber is realized by means of right translations, namely, $\forall p \in P$, $\exists a \in Q$: $p = \tilde{R}_a p_0$. This implies

$$\omega_p((\tilde{R}_a)_* X) = \mathsf{Ad}_a^{-1}(q_0)\omega_{p_0}(X), \qquad q_0 = \varphi(p_0)$$

It is easy to see that the obtained 1-form satisfies all conditions of Definition 3.1. ■

Let $\{U_\alpha, \Phi_\alpha\}$ be a local trivialization and let $\overline{\mathrm{Id}}$: $U_\alpha \mapsto U_\alpha \times Q$ by $x \mapsto (x, e)$. A trivialization $\Phi_\alpha$ defines a *canonical* section $\sigma_\alpha$ by the equation

$$\sigma_\alpha = \Phi_\alpha^{-1} \circ \overline{\mathrm{Id}}$$

and vice versa. We denote by $\omega_C$ a canonical $\mathsf{Ad}$-form.

*Definition 3.3.* Let $\omega_\alpha = \sigma_\alpha^* \omega$, where $\omega$ is the connection form. The form $\omega_\alpha$ on $U_\alpha$ is called the *connection form in the local trivialization* $\{U_\alpha, \Phi_\alpha\}$.

*Theorem 3.2.* At $U_\alpha \cap U_\beta$, local connection forms $\omega_\alpha$ and $\omega_\beta$, corresponding to the same connection $\omega$ on $P$, are related by

$$\omega_\beta = \mathsf{Ad}^{-1}_{q_{\alpha\beta}}(q_{\beta\alpha})\omega_\alpha + l_{(q_{\beta\alpha}, q_{\alpha\beta})^*}\theta_{\alpha\beta} \tag{9}$$

where $q_{\alpha\beta}$ are transition functions, and $\theta_{\alpha\beta} = q^*_{\alpha\beta}\omega_C$ denotes the pullback on $U_\alpha \cap U_\beta$ of the canonical 1-form $\omega_C$ on $Q$. Vice versa, for any set of the local forms $\{\omega_\alpha\}$ satisfying (9), there exists the unique connection form $\omega$ on $P$ generating this family of the local forms, namely, $\omega_\alpha = \sigma^*_\alpha \omega$, $\forall \alpha$.

*Proof.* 1. The direct theorem. Let $x \in U_\alpha \cap U_\beta$. Applying (3), we obtain $\sigma_\beta(x) = \tilde{R}_{q_{\alpha\beta}}\sigma_\alpha(x)$, $\forall x \in U_\alpha \cap U_\beta$. The map $(\sigma_\beta(x))_*$ transforms any vector $X \in T_x(U_\alpha \cap U_\beta)$ into $(\sigma_\beta)_* X \in T_{\sigma_\beta(x)}(P)$. Using the Leibniz formula (Kobayashi and Nomizu, 1963, Ch. I), we get

$$(\sigma_\beta)_* = (\tilde{R}_{q_{\alpha\beta}})_*(\sigma_\alpha)_* + (L_{q_{\beta\alpha}})_*(q_{\alpha\beta})_*$$

where $q_{\beta\alpha} := \varphi_\beta(\sigma_\alpha)$. Applying $\omega$ to both sides of this relation, we find

$$\begin{aligned} \omega_\beta(X) := \omega((\sigma_\beta)_* X) &= \omega((\tilde{R}_{q_{\alpha\beta}})_*(\sigma_\alpha)_* X) + \omega((L_{q_{\beta\alpha}})_*(q_{\alpha\beta})_* X) \\ &= \mathsf{Ad}^{-1}_{q_{\alpha\beta}}(q_{\beta\alpha})\omega_\alpha(X) + l_{(q_{\beta\alpha}, q_{\alpha\beta}(^*}(q^*_{\alpha\beta}\omega_C)(X) \\ &= \mathsf{Ad}^{-1}_{q_{\alpha\beta}}(q_{\beta\alpha})\omega_\alpha(X) + l_{(q_{\beta\alpha}, q_{\alpha\beta})^*}\theta_{\alpha\beta}\omega_C(X) \end{aligned}$$

2. The inverse theorem. Let us define the 1-form $\tilde{\omega}$ as follows:

$$\tilde{\omega} = \mathrm{Ad}_{q_\alpha}^{-1}(e)(\pi^*\omega_\alpha) + q_\alpha^*\omega_C, \qquad q_\alpha := \varphi_\alpha(p) \tag{10}$$

Let $X \in T_u(P)$ be an arbitrary vector and $u = \sigma_\alpha(\pi(x))$. Decompose $X$ into horizontal $Y$ and vertical $Z$ components:

$$X = Y + Z, \qquad Y = (\sigma_\alpha)_*(\pi_* X), \qquad \pi_* Z = 0$$

This implies

$$\begin{aligned}\tilde{\omega}(X) &= \mathrm{Ad}_{q_\alpha}^{-1}(e)\omega_\alpha(\pi_* X) + (q_\alpha^*\omega_C)(X)\\ &= \mathrm{Ad}_{q_\alpha}^{-1}(e)\omega((\sigma_\alpha)_*\pi_* X) + \omega_C((q_\alpha)_* X)\\ &=(\tilde{R}_{q_\alpha}^*\sigma_\alpha^*)\omega(\pi_* Y) + \omega_C((q_\alpha)_* Z)\\ &= \omega((\sigma_\alpha)_*\pi_* Y) + \omega_C((q_\alpha)_* Z)\\ &= \omega(Y) + \hat{Z} = \omega(Y) + \omega(Z) = \omega(X)\end{aligned}$$

and one sees that $\tilde{\omega} = \omega$ at any point of the section $\sigma_\alpha$. So these forms are transformed in the same way under the right translations and therefore coincide on $\pi^{-1}(U)$. ■

*Corollary 3.1.* For arbitrary sections $\sigma_1$ and $\sigma_2$ such that $\sigma_2 = R_q\sigma_1$ and $\omega_1 = \sigma_1^*\omega$, $\omega_2 = \sigma_2^*\omega$, the following relation holds:

$$\omega_2 = \mathrm{Ad}_q^{-1}(q_1)\omega_1 + l_{(q_1,q))*}(q^*\omega_C)$$

where $q_1 := \varphi(\sigma_1)$.

*Remark 3.1.* The local form $\sigma^*\omega$ is called a *gauge potential 1-form* in the physics literature.

## 3.1. Covariant Derivative. Curvature Form

Let $\{x^\mu, y^i\}$ be a local coordinate system in the neighborhood $\pi^{-1}(U_\alpha)$; $x^\mu$ are coordinates in $U_\alpha \in M$ and $y^i$ are coordinates in the fiber. Locally $\pi^{-1}(U_\alpha)$ can be presented as a direct production $U_\alpha \times Q$. The connection form can written in the form $\omega = \omega^i L_i$, with $\{L_i\}$ being the basis of left fundamental fields and $\{\omega^i\}$ the basis of 1-forms. Taking into account (10), we find that in the coordinates $\{x^\mu, y^i\}$,

$$\omega^i = (\mathrm{Ad}_y^{-1}(e))^i_j A^j_\mu(x)\, dx^\mu + \omega^i{}_j(y)\, dy^j \tag{11}$$

where $A^i_\mu(x)\, dx^\mu = \pi^*(\omega^i)$ and $\omega^i{}_j(y)\, dy^j = (L_*^{-1})^i_j\, dy^j$.

*Definition 3.4.* A *covariant derivative* $D_\mu$ in the principal $Q$-bundle is defined as follows:

$$D_\mu = \partial_\mu - A^i_\mu(x)\overline{L}_i$$

where the $\overline{L}_i = (R_*)^j_i \partial/\partial y^j$ are generators of the left translations (right quasiinvariant vector fields).

*Comment 3.1.* It is easy to show that $\omega(D_\mu) = 0$. Indeed, (11) implies

$$\omega(D_\mu) = [(\mathsf{Ad}_y^{-1}(e))^i_j A^j_\mu(x) - \omega^i{}_p A^j_\mu (R_*)^p_j]L_i$$

Noting that $\omega^i{}_p = (L_*^{-1})^i_p$, we obtain $\omega^i{}_p(R_*)^p_j = (\mathsf{Ad}_y^{-1}(e))^i_j$, and hence $\omega(D_\mu) = 0$.

Let us compute the commutator $[D_\mu, D_\nu]$:

$$[D_\mu, D_\nu] = (\partial_\nu A^i_\mu - \partial_\mu A^i_\nu)\overline{L}_i + A^i_\mu A^j_\nu[\overline{L}_i, \overline{L}_j]$$

Introducing $[\overline{L}_i, \overline{L}_j] = C^p_{ij}(y)\overline{L}_p$, we get

$$[D_\mu, D_\nu] = -F^i{}_{\mu\nu}\overline{L}_i, \qquad F^i{}_{\mu\nu} := \partial_\mu A^i_\nu - \partial_\nu A^i_\mu - A^j_\mu A^p_\nu C^i_{jp}$$

*Definition 3.5.* Let $\Psi$ be a vector-valued $r$-form in the principal $Q$-bundle. An $(r + 1)$-form $D\Psi$ defined by

$$D\Psi(X_1, X_2, \ldots, X_{r+1}) = d\Psi(horX_1, \ldots, horX_{r+1})$$

is called a *covariant differential* of the form $\Psi$.

*Definition 3.6.* A vector-valued 2-form $\Omega(X, Y)$ defined as

$$\Omega(X, Y) = D\omega(X, Y) = d\omega(horX, horY)$$

where $\omega$ is a connection form, is called a *curvature form*.

*Lemma. 3.1.* Let $X$, $Y$ be horizontal fields; then the following relation holds:

$$\omega([X, Y]) = -2\Omega(X, Y)$$

*Proof.* Applying the exterior differentiation to 1-form $\omega$, we obtain

$$2d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

As $X, Y \in \mathcal{H}_p$, then $\omega(X) = \omega(Y) = 0$. This implies $\omega([X, Y]) = -2\Omega(X, Y)$. ■

*Corollary 3.2.* The curvature form can be defined as

$$\Omega(X, Y) = -\tfrac{1}{2}\omega([hor\overline{Y}, hor\overline{Y}])$$

where $\overline{X}$, $\overline{Y}$ are any continuations of the vectors $X, Y \in T_p(P)$, respectively.

*Corollary 3.3.* The two-form $\Omega$ is an $\mathsf{Ad}$-form and is transformed under the right translations as

$$(\tilde{R}_a^* \Omega)(X, Y) = \mathsf{Ad}_a^{-1}(q)\Omega|_p(X, Y), \qquad \text{where} \quad q = \varphi(p)$$

*Theorem 3.3.* The curvature form $\Omega$ satisfies the structure equation

$$\Omega = d\omega + \omega \wedge \omega \tag{12}$$

*Proof.* To prove (12), one needs to consider all possible pairs $X$, $Y$. The same idea is used in the usual fiber bundle theory for proving that the curvature satisfies the structure equation (Kobayashi and Nomizu, 1963). ■

*Comment 3.2.* Choosing the family of the local sections $\sigma_\alpha$ associated with the trivialization $U_\alpha$, $\Phi_\alpha$, and taking into account that $\varphi_\alpha(\sigma_\alpha) = e$, where $\varphi_\alpha$ is the restriction of $\Phi_\alpha$ on $\pi^{-1}(U_\alpha)$, we obtain

$$\pi^*\Omega|_\alpha = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

where $F_{\mu\nu} = F^i_{\mu\nu}\hat{L}_j$ is introduced. Since $\Omega$ is an $\mathsf{Ad}$-form, the following transformation law holds:

$$\Omega|_\beta = \mathsf{Ad}^{-1}_{q_{\alpha\beta}}(q_{\beta\alpha})\Omega|_\alpha$$

*Theorem 3.4.* Bianchi identity: $D\Omega = 0$.

*Proof.* It is sufficient to show that $d\Omega(X, Y, Z) = 0$ if $X$, $Y$, $Z$ are horizontal vector fields. Applying the exterior derivative to (12), we obtain $d\Omega(X, Y, Z) = 0$ if $X$, $Y$, $Z$ are horizontal vector fields. ■

## 4. NONASSOCIATIVE GAUGE THEORY ON A PRINCIPAL LOOP *Q*-BUNDLE

Field theory is formulated as living not on spacetime $M$, but on the principal loop $Q$-bundle $P$. Choosing a natural coordinate system $(x, q)$ on $P$, where $x \in M$ and $q \in Q$, the Lagrangian $\mathscr{L}_g$ of the free gauge field is as follows:

$$\mathscr{L}_g = -\tfrac{1}{4}\langle F_{\mu\nu}, F^{\mu\nu}\rangle$$

where $\langle\ ,\ \rangle$ denotes the *Ad*-invariant scalar product on $Q$.

Fields are functions on $P$. With $\Psi(x) \in M$, we relate a function $\tilde{\Psi}(x, q)$ [lift of the function $\Psi(x)$] on the principal $Q$-bundle in the following way. We assume that $\tilde{\Psi}(x, q)$ transforms under a finite gauge transformation $U(q)$ of nonassociative representations of the loop $Q$ (Nesterov and Stepanenko, 1986; Nesterov, 1989) according to the inverse rule

$$\tilde{\Psi}(x, q' \cdot q) = U(q^{-1})\tilde{\Psi}(x, q'), \qquad q \in Q$$

Then the lift $\tilde{\Psi}(x, q)$ of the function $\Psi(x, q)$ is defined as follows:

$$\tilde{\Psi}(x, q) = U(q^{-1})\tilde{\Psi}(x, e)$$

where $\tilde{\Psi}(x, e) := \Psi(x)$, and $e \in Q$ is a neutral element.

The map of the element of quasialgebra $X \in \mathfrak{q}$ is defined by

$$X\tilde{\Psi}(x, q) = \frac{d}{dt}\tilde{\Psi}(x, q \cdot e^{-tX})\Big|_{t=0}$$

where $e^{-tX}$ is an exponential mapping on $Q$. This implies that the covariant derivative $\tilde{D}_\mu$ is given by

$$\tilde{D}\tilde{\Psi} = (\partial_\mu + \tilde{A}_\mu(x, q))\tilde{\Psi}\, dx^\mu = \tilde{D}_\mu\tilde{\Psi}\, dx^\mu$$

where we set $\tilde{A}_\mu = U(q)_*|_{q=e}\mathsf{Ad}_q^{-1}(e)A_\mu(x)$. The nonassociative gauge-invariant Lagrangian of matter has the structure

$$\mathcal{L}_m = \mathcal{L}(\tilde{\Psi}, \tilde{D}_\mu\tilde{\Psi}, x)$$

and describes a matter field $\Psi$ minimally coupling with the gauge field $A_\mu$.

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